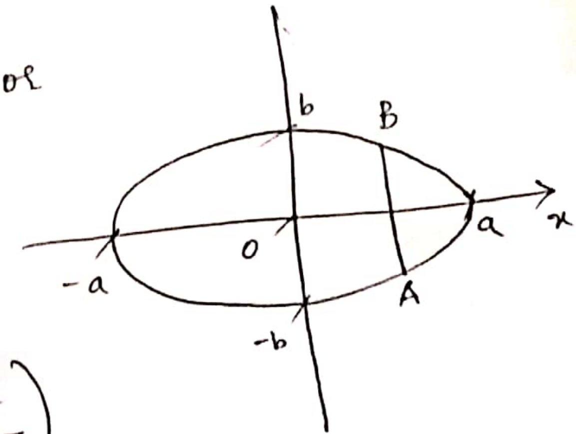
1. Evaluate $\iint_R (x+y)^2\, dx\, dy$, where R is the region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



Sol^n 'x' varies from $-a$ to a and for each x, y varies from a point A to a point B, where A and B are shown in the fig

We have $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y^2=b^2\left(1-\frac{x^2}{a^2}\right)$

(or) $y=\pm b\sqrt{1-\frac{x^2}{a^2}}$

For the point A: $y_1=-b\sqrt{1-\frac{x^2}{a^2}}$

B: $y_2=b\sqrt{1-\frac{x^2}{a^2}}$

$$\iint_R (x+y)^2\,dx\,dy=\int_{-a}^{a}\int_{y_1}^{y_2}(x+y)^2\,dy\,dx=\int_{-a}^{a}\left\{\int_{y_1}^{y_2}(x^2+2xy+y^2)\,dy\right\}dx$$

$$=\int_{-a}^{a}\left\{x^2y+\frac{2xy^2}{2}+\frac{y^3}{3}\right\}_{y_1}^{y_2}dx$$

$$=\int_{-a}^{a}\left\{x^2(y_2-y_1)+x(y_2^2-y_1^2)+\tfrac{1}{3}(y_2^3-y_1^3)\right\}dx$$

$$=\int_{-a}^{a}\left\{x^2\left(b\sqrt{1-\frac{x^2}{a^2}}+b\sqrt{1-\frac{x^2}{a^2}}\right)+x\left(b^2\left(1-\frac{x^2}{a^2}\right)\right)-b^2\left(1-\frac{x^2}{a^2}\right)\right)\right.$$
$$\left.+\tfrac{1}{3}\left[b^3\left(1-\frac{x^2}{a^2}\right)^{3/2}+b^3\left(1-\frac{x^2}{a^2}\right)^{3/2}\right]\right\}dx$$

Scanned with CamScanner

$$= \int_{-a}^{a} \left\{ 2bx^2\left(1-\frac{x^2}{a^2}\right)^{1/2} + \frac{2b^3}{3}\left(1-\frac{x^2}{a^2}\right)^{3/2} \right\} dx$$

$$= 4\int_{0}^{a} \left\{ bx^2\left(1-\frac{x^2}{a^2}\right)^{1/2} + \frac{b^3}{3}\left(1-\frac{x^2}{a^2}\right)^{3/2} \right\} dx \qquad \text{Put } x = a\sin\theta,\ dx = a\cos\theta\, d\theta$$

$$= 4\int_{0}^{\pi/2} \left\{ ba^2\sin^2\theta\cos\theta + \frac{b^3}{3}\cos^3\theta \right\} a\cos\theta\, d\theta$$

$$= 4ba^3\int_{0}^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta + \frac{4}{3}ab^3\int_{0}^{\pi/2} \cos^4\theta\, d\theta$$

$$= 4ba^3\left(\pi/16\right) + \frac{4}{3}ab^3\left(\frac{3\pi}{16}\right) = \frac{\pi}{4}ab(a^2+b^2)$$

Evaluate the following integrals.

1. $\displaystyle\int_{1}^{4}\int_{0}^{\sqrt{4-x}} xy\,dy\,dx$

2. $\displaystyle\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}} y^3\,dy\,dx$

3. $\displaystyle\int_{0}^{2}\int_{x^2}^{2x} (2x+3y)\,dy\,dx$

4. $\displaystyle\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}} x^2y\,dy\,dx$

5. $\displaystyle\int_{0}^{1}\int_{0}^{x} e^{x/y}\,dy\,dx$

6. $\displaystyle\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}} dx\,dy$

7. $\displaystyle\int_{0}^{2a}\int_{0}^{\sqrt{2ax-x^2}} xy\,dy\,dx$

8. $\displaystyle\int_{0}^{1}\int_{y^2}^{y} (1+xy^2)\,dx\,dy$

9. Evaluate $\iint_R (x+y)\,dx\,dy$ where R is the triangular region bounded by $y=2x$, $y=x/2$ & $y=3-x$

10. $\iint_R \cancel{(x+y)}\, xy\,dx\,dy$ where R is the triangular region with vertices $(-6,2)$, $(-1,3)$, $(9,-7)$

12. If A is the area bounded by the circle $x^2+y^2=a^2$ in the I quadrant. S.T

$$\int_A xy\,dA = \frac{a^2}{8}$$

13. Evaluate $\iint_A \sqrt{4x^2-y^2}\,dx\,dy$ where A is the region bounded by the lines $y=0$, $y=x$, $x=1$.

14. Evaluate $\iint_R x^2y\,dx\,dy$ where R is the region bounded by x- axis and the lines $y=x$ and $y=2-x$.

15. If R is the region bounded by the lines $x=2$, $y=1$ and the parabola $y=x^2$ show that

$$\iint_R (x^2+y^2)\,dx\,dy = \frac{1006}{105}$$

Change of order of Integration

1) When the limits are constants, then the order of integration is immaterial

i.e., $\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$. Here

we have to Keep it is mind that the limits of 'x' are to be used for x and those of 'y' used for 'y' only.

2) When the limits of integration are Variables, on changing the order of integration, the limits of integration change. To find the new limits, a rough sketch of the region of integration is essential. This helps in fixing the new limits of integration connected with Some of the problems which seem to be double integrals, can be made easy to complicated handle by a change in the order of integration.

Problems

1. Change the order of integration in the integral $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy$ and hence evaluate.

$\underline{\text{sol}^n}$ The region of integration is bounded by $y=0$, $y=3$ and the Curve $x=1$ and $x=\sqrt{4-y}$.

i.e., $y=0, y=3$ and the curves $x=1$ and $x^2=4-y$.

Now to change the order of integration, we have to make the limits of 'x' independent of y. Solving $x=1$ and $x^2=4-y$ we get

$B=(1,3)$, $C=(2,0)$

Now 'x' varies from 1 to 2 and y varies from 0 to $4-x^2$.

Hence,

$$\int_0^3\int_1^{\sqrt{4-y}}(x+y)\,dx\,dy=\int_1^2\int_0^{4-x^2}(x+y)\,dy\,dx=\int_1^2\left[\int_0^{4-x^2}(x+y)\,dy\right]dx$$

$$=\int_1^2\left[xy+\frac{y^2}{2}\right]_0^{4-x^2}dx=\int_1^2\left[x(4-x^2)+\left(\frac{4-x^2}{2}\right)^2\right]dx$$

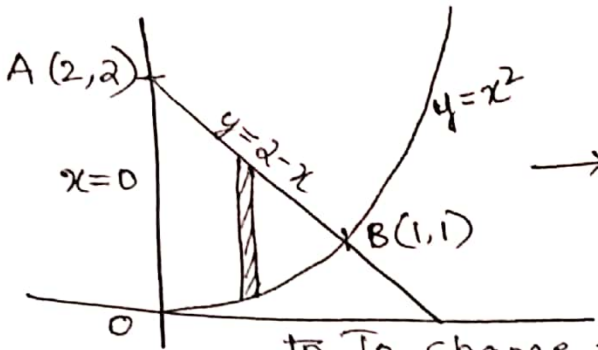
$$=\int_1^2\left[4x-x^3+\frac{16+x^4-8x^2}{2}\right]dx=\int_1^2\left[4x-x^3+8+\frac{x^4}{2}-4x^2\right]dx$$

$$=\left[4\frac{x^2}{2}-\frac{x^4}{4}+8x+\frac{x^5}{10}-\frac{4x^3}{3}\right]_1^2=\frac{241}{60}$$

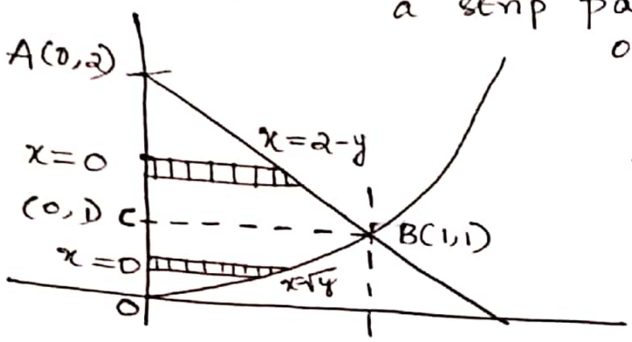
(1)

2) Evaluate $\int_0^1 \int_{x^2}^{2-x} xy \, dx dy$ by changing the order of integration

sol^n The region of integration is shown in the figure bounded by parabola $y=x^2$ & the line $y=2-x$. The point of intersection of the parabola $y=x^2$ and the line $y=2-x$ is $B=(1,1)$.



→ Given strip parallel to y-axis.



To change the order of integrat'n, we have to take a strip parallel to x-axis. i.e., in the area bounded by OBC & OAB, i.e., area ABC.

The limits of 'x' in area OBC, are $x=0$ & $x=\sqrt{y}$

The limits of 'x' in are ABC, are $x=0$ & $x=2-y$.

So, the integral is

$$= \int_0^1 \int_0^{\sqrt{y}} xy \, dx dy + \int_1^2 \int_0^{2-y} xy \, dx dy = \int_0^1 y dy \left(\frac{x^2}{2}\right)_0^{\sqrt{y}} + \int_1^2 y dy \left(\frac{x^2}{2}\right)_0^{2-y}$$

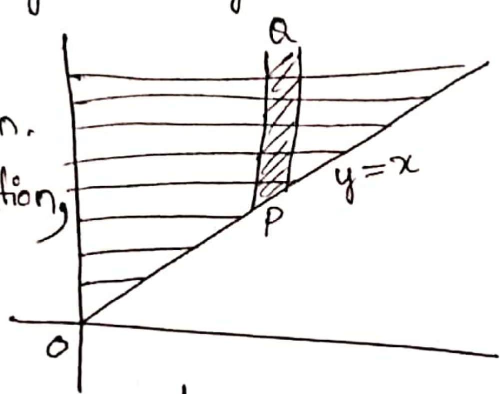
$$= \frac{1}{2} \int_0^1 y^2 dy + \frac{1}{2} \int_1^2 y(2-y)^2 dy = \frac{1}{2}\left(\frac{y^3}{3}\right)_0^1 + \frac{1}{2}\int_1^2 (4y-4y^2+y^3) dy$$

$$= \frac{1}{6} + \frac{1}{2}\left[2y^2 - \frac{4}{3}y^3 + \frac{y^4}{4}\right]_1^2 = \frac{1}{6} + \frac{1}{2}\left[\left(8 - \frac{32}{3} + 4\right) - \left(2 - \frac{4}{3} + \frac{1}{4}\right)\right]$$

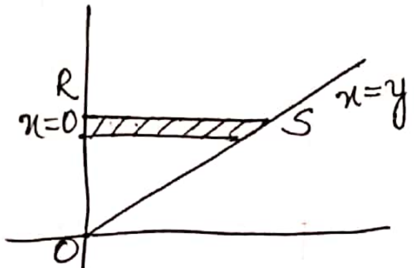
$$= \frac{3}{8}$$

③ Evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dx\,dy$ by changing the order of integration.

Sol^n The elementary strip extends from $y=x$ to $y=\infty$ & $x=0$ to $x=\infty$. The shaded portion is the region of integration.



On changing the order of integration, we first integrate w.r.t 'x' along the horizontal strip RS which extends from $x=0$ to $x=y$.



In this, we integrate w.r.t 'y' from $y=0$ to $y=\infty$.

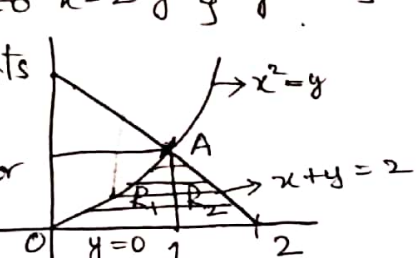
$$= \int_0^\infty \int_0^y \frac{e^{-y}}{y}\,dx\,dy = \int_0^\infty \frac{e^{-y}}{y}\,dy\,[x]_0^y$$

$$= \int_0^\infty y\frac{e^{-y}}{y}\,dy = \left(\frac{e^{-y}}{-1}\right)_0^\infty = -\left(\frac{1}{e^y}\right)_0^\infty = -\left(\frac{1}{\infty}-1\right) = 1.$$

④ Evalnate $\int_0^1 \int_{\sqrt{y}}^{2-y} xy\,dx\,dy$ by changing the order of integration.

sol^n Given strip are from $x=\sqrt{y}$ & to $x=2-y$ & $y=0$ & $y=1$.

We observe that 'R' is made up of 2 parts R_1 & R_2.

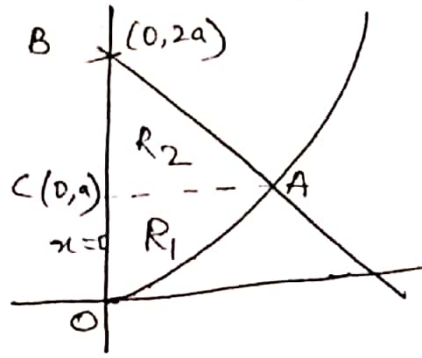


In R_1, x varies from $x=0$ to $x=1$ & for each x, y varies from $y=0$ to $y=x^2$.

In R_2, 'x' varies from $x=1$ to $x=2$ & for each x, y varies from $y=0$ to $y=2-x$.

$$= \int_{x=0}^{1}\int_{y=0}^{x^2} xy\,dy\,dx + \int_{x=1}^{2}\int_{y=0}^{2-x} xy\,dy\,dx$$

5 Evaluate $\int_0^a \int_{x^2/a}^{2a-x} xy\,dy\,dx$ by changing the order of integration



Soln Given strip varies from $x=0$ to $x=a$ and for each x, 'y' varies from $y=\frac{x^2}{a}$ to $y=2a-x$. The region of integration is shown in the fig. Observe that 'R' is made up of 2 parts R_1 & R_2.

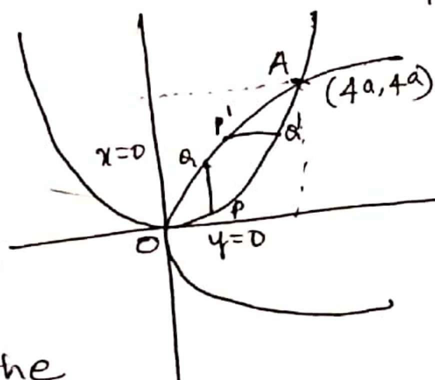
In R_1, y varies from 0 to a for each y, 'x' varies from $x=0$ to $x=\sqrt{ay}$. In R_2, 'y' varies from $y=a$ to $y=2a$ for each y, 'x' varies from $x=0$ to $x=2a-y$.

$$= \int_{y=0}^{a} \int_{x=0}^{\sqrt{ay}} xy\,dx\,dy + \int_{y=a}^{2a} \int_{x=0}^{2a-y} xy\,dx\,dy$$

6 Change the order of integration & evaluate $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$

Soln ~~We have to~~

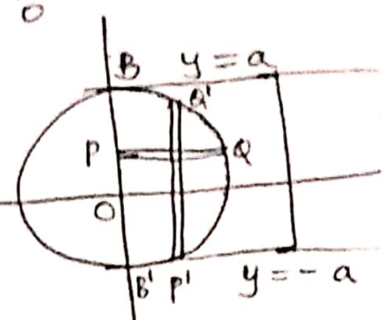
Here, integration is first performed along the vertical strip PQ. For changing the order of integration, we divide the region of integration OPAQO into horizontal strip P'Q' which extends from P' on the parabola $y^2 = 4ax$ i.e., $x = y^2/4a$ to Q' on the parabola $x^2 = 4ay$ i.e., $x = 2\sqrt{ay}$



$$= \int_0^{4a} \int_{y^2/4a}^{2\sqrt{ay}} dx\,dy$$

7) Change the order of integration $\int_{-a}^{a}\int_{0}^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$

Sol^n from the given limits of integration it is clear that I integrat'n is w.r.t 'x' between the limits $x=0$ to $x=\sqrt{a^2-y^2}$ & then w.r.t 'y' between the limits $y=-a$ to a. So, integration is performed first along the horizontal strip from 'P' on $x=0$ to the point Q on the Circle $x=\sqrt{a^2-y^2}$ (i.e., $x^2=a^2-y^2$ (or) $x^2+y^2=a^2$).

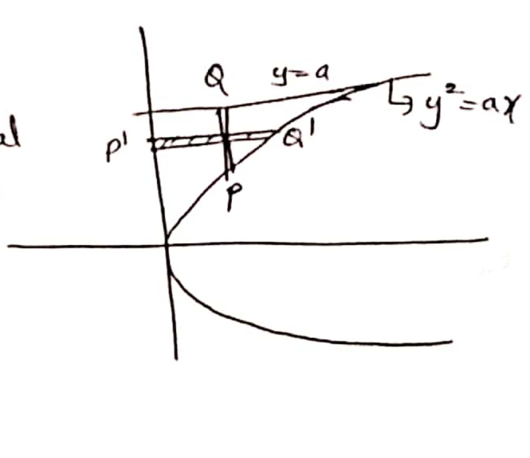


To change the order of integration, divide the region of integrat'n $B'AQBP$ in to vertical strip $P'Q'$ which extends from P' on the circle $y=-\sqrt{a^2-x^2}$ to Q' on the Circle $y=\sqrt{a^2-x^2}$.

The strip slides from $x=0$ to $x=a$.

$$\therefore \int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} f(x,y)\,dx\,dy = \int_{0}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y)\,dy\,dx$$

8) Change the order of integration & hence evaluate $\int_{0}^{a}\int_{\sqrt{ax}}^{a} \frac{y^2\,dx\,dy}{\sqrt{y^4-a^2x^2}}$

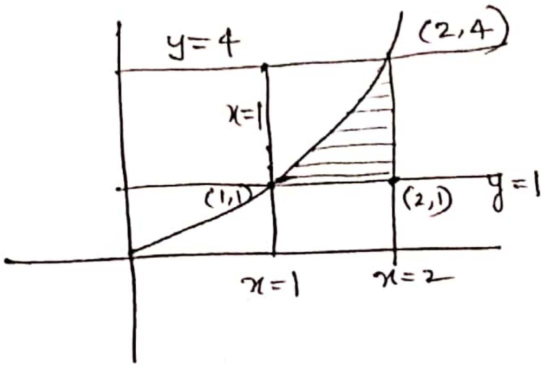


Sol^n I integration is performed along vertical strip PQ. On changing the order of integration we integrate along $P'Q'$.

$$= \int_{0}^{a}\int_{0}^{y^2/a} \frac{y^2\,dy}{\sqrt{y^4-a^2x^2}}\,dx$$

(3)

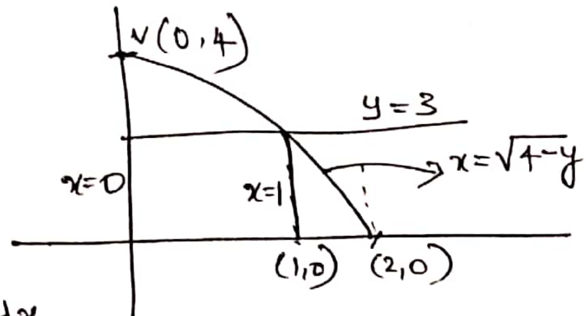
(9) Change the order of integration and hence evaluate $\int_1^2 \int_1^{x^2} (x^2+y^2)\,dy\,dx$



Sol^n: On changing the order of integrat'n 'y' varies from 1 to 4 and 'x' varies from $\sqrt{y}$ to 2.

$$\therefore \; = \int_1^4 \int_{\sqrt{y}}^2 (x^2+y^2)\,dx\,dy$$

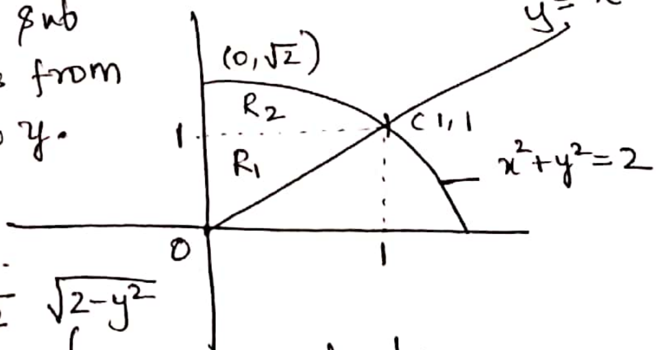
(10) Change the order of integration and hence evaluate $\int_0^3 \int_0^{\sqrt{4-y}} (x+y)\,dx\,dy$



Sol^n: Changing the order,

$$= \int_{x=0}^{1} \int_{y=0}^{3} (x+y)\,dy\,dx + \int_{x=1}^{2} \int_{y=0}^{4-x^2} (x+y)\,dy\,dx$$

(11) Change the order of integration in $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$



Sol^n: The region R is made up of 2 sub regions R_1 & R_2. In R_1 'y' varies from 0 to 1 & 'x' varies from 0 to y.

In R_2, 'y' varies from 1 to $\sqrt{2}$ & 'x' varies from 0 to $\sqrt{2-y^2}$.

$$= \int_0^1 \int_0^y \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy$$

Problems

1. $$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$$

2. $$\int_0^{1}\int_{x}^{1} (x^2+y^2)\,dy\,dx$$

3. $$\int_0^{1}\int_{x}^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$$

4. $$\int_0^{\infty}\int_0^{x} x\,e^{-x^2/y}\,dy\,dx$$

5. $$\int_0^{a}\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx$$

6. $$\int_0^{1}\int_{\sqrt{y}}^{1} dx\,dy$$

7. $$\int_0^{1}\int_0^{\sqrt{1-x^2}} y^2\,dy\,dx$$

8. $$\int_0^{1}\int_{y^2}^{\sqrt{y}} \frac{y}{x}\,e^{x}\,dx\,dy$$

9. $$\int_0^{1}\int_{y}^{\sqrt{y}} xy\,dx\,dy$$

10. $$\int_0^{2}\int_{y^2/4}^{3-y} (x^2+y^2)\,dx\,dy.$$